

# On central automorphisms of finite $p$ -groups

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## Abstract

We characterize all finite  $p$ -groups  $G$  of order  $p^n$  ( $n \leq 6$ ), where  $p$  is a prime for  $n \leq 5$  and an odd prime for  $n = 6$ , such that the center of the inner automorphism group of  $G$  is equal to the group of central automorphisms of  $G$ .

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## 1 Introduction

Let  $G$  be a finite  $p$ -group. By  $Z(G)$ ,  $\gamma_2(G)$ ,  $\text{Aut}(G)$  and  $\text{Inn}(G)$ , we denote the center, the commutator subgroup, the group of all automorphisms and the group of all inner automorphisms of  $G$  respectively. An automorphism  $\alpha$  of  $G$  is called a central automorphism if  $\alpha$  commutes with every inner automorphism, or equivalently, if  $x^{-1}\alpha(x)$  lies in the centre of  $G$  for all  $x$  in  $G$ . The central automorphisms fix the commutator subgroup  $\gamma_2(G)$  of  $G$  pointwise and form a normal subgroup  $\text{Aut}_z(G)$  of  $\text{Aut}(G)$ . The group  $\text{Aut}_z(G)$  always lies between  $Z(\text{Inn}(G))$  and  $\text{Aut}(G)$ . Non-abelian  $p$ -groups in which all automorphisms are central have been well studied (see [6] for references). Curran and McCaughan [6] have characterized those finite  $p$ -groups for which  $\text{Aut}_z(G) = \text{Inn}(G)$ . They in fact proved that  $\text{Aut}_z(G) = \text{Inn}(G)$  if and only if  $\gamma_2(G) = Z(G)$  and  $Z(G)$  is cyclic. Curran [5] considered the case when  $\text{Aut}_z(G)$  is minimum possible, that is, when  $\text{Aut}_z(G) = Z(\text{Inn}(G))$ . He proved that for  $\text{Aut}_z(G)$  to be equal to  $Z(\text{Inn}(G))$ ,  $Z(G)$  must be contained in  $\gamma_2(G)$  and  $Z(\text{Inn}(G))$  must not be cyclic. These conditions are necessary but not sufficient because there are groups  $G$  for which  $\text{Aut}_z(G) \neq Z(\text{Inn}(G))$  even if  $Z(G)$  is contained in  $\gamma_2(G)$ .

or  $Z(\text{Inn}(G))$  is not cyclic [see section 4]. The present note is the result of an effort to find the sufficient conditions. Observe that if nilpotency class of  $G$  is 2, then  $Z(\text{Inn}(G)) = \text{Inn}(G)$  and thus  $\text{Aut}_z(G) = Z(\text{Inn}(G))$  if and only if  $\gamma_2(G) = Z(G)$  and  $Z(G)$  is cyclic. Also observe that if  $G$  is of maximal class, then  $|Z(\text{Inn}(G))| = p$  and by [5]  $\text{Aut}_z(G) > Z(\text{Inn}(G))$ . Therefore, to characterize all finite  $p$ -groups  $G$  for which  $\text{Aut}_z(G) = Z(\text{Inn}(G))$ , we can assume that nilpotency class of  $G$  is bigger than 2 and  $G$  is not of maximal class. Of course in such a case  $|G| \geq p^5$ . In section 3, we prove two theorems which characterize all such groups of order  $p^5$ , where  $p$  is any prime, and of order  $p^6$ , where  $p$  is an odd prime. In section 4, we find all such groups from the list of groups of order  $p^n$ , where  $p$  is an odd prime and  $4 \leq n \leq 6$ , ordered into isoclinism families by James [7].

## 2 Preliminaries

By  $\text{Hom}(G, A)$ , we denote the group of all homomorphisms of  $G$  into an abelian group  $A$  and by  $C_{p^n}$ , we denote the cyclic group of order  $p^n$ . The nilpotency class of  $G$  is denoted as  $cl(G)$  and by  $d(G)$  we denote the rank of  $G$ . For a  $p$ -group  $G$ , let  $\Omega_1(G) = \langle x \in G | x^p = 1 \rangle$  and  $\mathcal{U}_1(G) = \langle x^p | x \in G \rangle$ . A non-abelian group  $G$  that has no non-trivial abelian direct factor is said to be purely non-abelian. Observe that a group  $G$  is purely non-abelian if its center  $Z(G)$  is contained in the frattini subgroup  $\Phi(G)$ . A  $p$ -group  $G$  is said to be regular if for any two elements  $x, y$  in  $G$ , there is an element  $z$  in the commutator subgroup  $\gamma_2(H)$  of the subgroup  $H = \langle x, y \rangle$  of  $G$ , such that  $x^p y^p = (xy)^p z^p$ . In the following lemma we list two important properties of the groups  $G$  for which  $\text{Aut}_z(G) = Z(\text{Inn}(G))$ .

**Lemma 2.1** [5, Corollaries 3.7, 3.8] *Let  $G$  be a finite non-abelian  $p$ -group such that  $\text{Aut}_z(G) = Z(\text{Inn}(G))$ . Then  $Z(G) \leq \gamma_2(G)$  and  $Z(\text{Inn}(G))$  is not cyclic.*

The following well known results are also of our interest and will be referred to without any citation.

**Theorem 2.2** [1] *If  $G$  is a purely non-abelian finite group, then  $|\text{Aut}_z(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))|$ .*

**Lemma 2.3** *Let  $A, B$  and  $C$  be finite abelian groups. Then*

- (i)  $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$ .
- (ii)  $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$ .

**Lemma 2.4** *Let  $C_m$  and  $C_n$  be two cyclic groups of order  $m$  and  $n$  respectively. Then  $\text{Hom}(C_m, C_n) \cong C_d$ , where  $d$  is the greatest common divisor of  $m$  and  $n$ .*

**Lemma 2.5** *If  $G$  is a regular  $p$ -group, then for all  $x, y \in G$  and  $i, j \geq 0$ ,*

$$[x^{p^i}, y^{p^j}] = 1 \text{ if and only if } [x, y]^{p^{i+j}} = 1.$$

### 3 Characterization

In this section we characterize all groups of order  $p^5$ , where  $p$  is any prime, and all groups of order  $p^6$ , where  $p$  is an odd prime, such that  $\text{Aut}_z(G) = Z(\text{Inn}(G))$ . The characteristics of such groups turn up same in both the cases. We begin with the following simple lemma.

**Lemma 3.1** *Let  $G$  be a finite  $p$ -group of order  $p^4$  and rank 2. If  $|\gamma_2(G)| = p$ , then  $Z(G) = \Phi(G)$  and  $|Z(G)| = p^2$ .*

**Proof.** Observe that  $|\Phi(G)| = p^2$  and  $cl(G) = 2$ . Therefore  $\exp(G/Z(G)) = \exp(\gamma_2(G)) = p$  and hence  $Z(G) = \Phi(G)$ .

**Theorem 3.2** *Let  $G$  be a finite  $p$ -group such that  $|G| = p^5$  and  $cl(G) = 3$ . Then  $\text{Aut}_z(G) = Z(\text{Inn}(G))$  if and only if  $d(G) = 2$  and  $|Z(G)| = p$ .*

**Proof.** First suppose that  $d(G) = 2$  and  $|Z(G)| = p$ . Then  $G$  is purely non-abelian and thus  $|\text{Aut}_z(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))| = p^2$ . Since  $Z(\text{Inn}(G)) \leq \text{Aut}_z(G)$ ,  $|Z(\text{Inn}(G))| = p$  or  $p^2$ . If  $|Z(\text{Inn}(G))| = p$ , then  $G/Z(G)$  is a 2-generated non abelian group of order  $p^4$  having nilpotency class 2. Thus  $|Z(G/Z(G))| = |Z(\text{Inn}(G))| = p = \gamma_2(G/Z(G))$ , which is a contradiction to Lemma 3.1. Therefore  $|Z(\text{Inn}(G))| = p^2$  and hence  $\text{Aut}_z(G) = Z(\text{Inn}(G))$ .

Conversely suppose that  $\text{Aut}_z(G) = Z(\text{Inn}(G))$ . Then  $Z(G) < \gamma_2(G)$  by Lemma 2.1 and the fact that  $cl(G) = 3$ . Therefore  $G$  is purely non-abelian and  $\text{Inn}(G)$  is a non-abelian group of order  $p^4$ , because if  $|\text{Inn}(G)| = p^3$ , then  $|Z(\text{Inn}(G))| = p$  which is a contradiction to Lemma 2.1. Thus  $|Z(G)| = p$  and  $|Z(\text{Inn}(G))| = p^2$ . Then  $|\text{Aut}_z(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))| = |Z(\text{Inn}(G))| = p^2$  implies that  $d(G) = 2$ .

**Theorem 3.3** *Let  $G$  be a finite  $p$ -group,  $p$  an odd prime, such that  $|G| = p^6$  and  $cl(G) = 3$  or 4. Then  $\text{Aut}_z(G) = Z(\text{Inn}(G))$  if and only if  $d(G) = 2$  and  $|Z(G)| = p$ .*

**Proof.** If  $d(G) = 2$  and  $|Z(G)| = p$ , then as in above theorem,  $|\text{Aut}_z(G)| = p^2$ . We proceed to prove that  $|Z(\text{Inn}(G))| = p^2$ . On the contrary suppose that  $|Z(\text{Inn}(G))| = p$ . First suppose that  $cl(G) = 3$ . Let  $H = G/Z(G)$ . Since the nilpotency class of  $G$  is 3, the nilpotency class of  $H$  is 2. Thus  $\gamma_2(H) = Z(H) = Z(\text{Inn}(G))$  is of order  $p$ . This implies that the exponent of  $H/Z(H)$  is  $p$  and therefore  $H$  is an extra-special  $p$ -group, which is a contradiction to the fact that  $d(H) = 2$ . Next suppose that  $cl(G) = 4$ . Let  $H = G/Z(G)$  and let  $K = H/Z(H)$ . Then  $|K| = p^4$ ,  $cl(K) = 2$  and  $\exp(K/Z(K)) = \exp(\gamma_2(K))$ . Now  $|Z(K)| \neq p$  because if  $|Z(K)| = p$ , then  $d(K) = 3$ . Thus  $|Z(K)| = p^2$  and hence  $|\gamma_2(K)| = p$  [8, Theorem 2.1]. It then follows that  $|\gamma_2(H)| = p^2$ ,  $|\gamma_2(G)| = p^3$  and  $|\gamma_3(G)| = p^2$ . Let  $\{x, y\}$  be a minimal generating set for  $H$ . If  $\gamma_2(H)$  is elementary abelian, then

$$[x^p, y] = [x, y]^{x^{p-1}} [x, y]^{x^{p-2}} \dots [x, y]^x [x, y] = [x, y]^p [x, y, x]^{p(p-1)/2} = 1.$$

Thus  $x^p$  and similarly  $y^p$  is in  $Z(H) < \gamma_2(H)$ . Therefore  $\Phi(H) = \gamma_2(H)$ , a contradiction to the fact that  $d(H) = 2$ . If  $\gamma_2(H)$  is cyclic, then  $|\Phi(\gamma_2(H))| = p$ . But  $\Phi(\gamma_2(H)) = \Phi(\gamma_2(G)/Z(G)) = \Phi(\gamma_2(G))/Z(G)$ , because  $Z(G)$  being of order  $p$  is contained in  $\Phi(\gamma_2(G))$ . This implies that  $|\Phi(\gamma_2(G))| = p^2$ . It then follows that  $\gamma_2(G)$  is cyclic and therefore  $G$  is regular. Suppose  $G/\gamma_2(G) \approx C_{p^2} \times C_p \approx \langle x\gamma_2(G) \rangle \times \langle y\gamma_2(G) \rangle$ , then  $\{x, y\}$  is a minimal generating set for  $G$ ,  $\gamma_2(G) = \langle [x, y] \rangle$  and  $x^{p^2}, y^p \in \gamma_2(G)$  but  $x^p \notin \gamma_2(G)$ . Since  $G$  is of class 4,  $\gamma_3(G) \leq Z_2(G)$ . But  $|Z_2(G)/Z(G)| = |Z(\text{Inn}(G))| = p = |Z(G)|$  implies that  $|Z_2(G)| = p^2 = |\gamma_3(G)|$  and thus  $\gamma_3(G) = Z_2(G)$ . If  $y^p \in Z_2(G) = \gamma_3(G)$ , then  $[y^p, x] \in Z(G)$  and since  $G$  is regular,  $1 = [y^p, x]^p = [y^{p^2}, x] = [y, x]^{p^2}$ , a contradiction to the fact that  $|\gamma_2(G)| = p^3$ . Therefore  $y^p \in \gamma_2(G) - \gamma_3(G)$ ,  $\gamma_2(G) = \langle y^p \rangle$  and  $|y| = p^4$ . Thus  $x^{p^2} \in \langle y \rangle$  and therefore  $[x^{p^2}, y] = [x, y]^{p^2} = 1$ , which is again a contradiction to the fact that  $|\gamma_2(G)| = p^3$ . Hence  $\text{Aut}_z(G) = Z(\text{Inn}(G))$  is of order  $p^2$ .

Conversely suppose that  $Z(\text{Inn}(G)) = \text{Aut}_z(G)$ . Then  $Z(G) < \gamma_2(G)$  by Lemma 2.1 and the fact that  $cl(G) > 2$ . Thus  $G$  is purely non-abelian and  $G/Z(G)$  is a non-abelian group of order at most  $p^5$ . Therefore  $p^2 \leq |Z(\text{Inn}(G))| \leq p^3$ , because  $Z(\text{Inn}(G))$  cannot be cyclic by Lemma 2.1. The possibility of  $G$  being rank 4 is immediately ruled out because if  $d(G) = 4$ , then

$$|\text{Aut}_z(G)| = |\text{Hom}(C_p \times C_p \times C_p \times C_p, C_p)| = p^4 \neq |Z(\text{Inn}(G))|.$$

If  $d(G) = 3$ , then  $|\text{Aut}_z(G)| \geq p^3$  and therefore by assumption,  $|\text{Aut}_z(G)| = |Z(\text{Inn}(G))| = p^3$ . Also  $|Z(G)| = p$ , for if  $|Z(G)| = p^2$ , then  $|Z(\text{Inn}(G))| = p^2$ . Let  $H = G/Z(G)$ , then  $|H/Z(H)| = p^2$  and hence  $|\gamma_2(H)| = p$ . It then follows that  $|\gamma_2(G)| = p^2$ ,  $|\gamma_3(G)| = p$  and therefore  $cl(G) = 3$ . Suppose that

$$G/\gamma_2(G) \approx C_p \times C_p \times C_{p^2} \approx \langle x\gamma_2(G) \rangle \times \langle y\gamma_2(G) \rangle \times \langle z\gamma_2(G) \rangle.$$

Then  $G = \langle x, y, z \rangle$  and  $x^p, y^p, z^{p^2} \in \gamma_2(G)$  but  $z^p \notin \gamma_2(G)$ . Since  $|G/Z_2(G)| = p^2$ , one of  $x, y$  and  $z$  lies in  $Z_2(G)$ . If  $z \in Z_2(G)$ , then  $z^p \in Z(G) \leq \gamma_2(G)$ , which is a contradiction. We can therefore, without any loss of generality, assume that  $x \in Z_2(G)$ . Then  $[x, y], [x, z] \in Z(G)$  and  $[z^p, x] = 1$ . If  $\gamma_2(G)$  is elementary abelian, then  $[z^p, y] = [z, y]^p [z, y, z]^{p(p-1)/2} = 1$ . Thus  $z^p \in Z(G)$ , again a contradiction. Now suppose that  $\gamma_2(G)$  is cyclic. Let  $M = \langle y, z, \Phi(G) \rangle$ , then  $M$  is a maximal subgroup of  $G$  and  $G = MZ_2(G)$ . It follows from [3, Theorem 1.3] that  $\gamma_i(G) = \gamma_i(M)$  for all  $i \geq 2$ . Since  $Z(G) \leq M$ ,  $Z(G) \leq Z(M)$  and  $C_G(M) = Z(M)$ . We prove that  $Z(G) = Z(M)$ . Since  $|M| = p^5$ , order of  $Z(M)$  lies between  $p$  and  $p^3$ . If  $|Z(M)| = p^2$  or  $p^3$ , then since  $\gamma_2(M) = \gamma_2(G)$  is cyclic, it follows from [2, Prop. 21.20] that

$$p^3 \geq |M/Z(M)| \geq |\gamma_2(M)|^2 = p^4,$$

a contradiction and thus  $|Z(M)| = p = |Z(G)|$ . This proves that  $Z(M) = Z(G)$ . Since  $\gamma_2(G)$  is cyclic,  $\Omega_1(\gamma_2(G)) = Z(G)$  and thus  $C_G(C_G(M)) = M$  [4, Theorem C]. Now from the facts that  $C_G(M) = Z(M)$ ,  $Z(M) = Z(G)$  and  $C_G(C_G(M)) = M$ , it follows that  $C_G(Z(G)) = M$ , a final contradiction to

$d(G) = 3$ . Hence  $d(G)=2$  and therefore, because of assumption,  $|Z(G)| = p$  or  $p^2$ . We prove that  $|Z(G)| = p$ . If  $|Z(G)| = p^2$ , then  $|Z(\text{Inn}(G))| = p^2$  and therefore  $|\gamma_2(G)/Z(G)| = p$ . This implies that  $|\gamma_2(G)| = p^3$  and  $G/\gamma_2(G) \approx C_{p^2} \times C_p$ . Thus  $|\text{Aut}_z(G)| = |\text{Hom}(C_{p^2} \times C_p, Z(G))| \geq p^3 > |Z(\text{Inn}(G))|$ , a contradiction to the assumption. This proves the “if” part of the theorem.

## 4 Application

In this section, we use the classification of all groups of order  $p^n$ , where  $p$  is an odd prime and  $5 \leq n \leq 6$ , given by James [7]. As an application of our results, we find those groups  $G$  of order  $p^5$  and  $p^6$  for which  $\text{Aut}_z(G) = Z(\text{Inn}(G))$ . We shall mainly use the informations of these groups given in §4.1 and presentations of these groups given in §4.5 and §4.6 of [7].

**Theorem 4.1** *If  $|G| = p^5$  and  $cl(G) = 3$ , then  $\text{Aut}_z(G) = Z(\text{Inn}(G))$  if and only if  $G$  is isomorphic to  $\Phi_8(32)$ .*

**Proof.** There are only 2 isoclinism families *viz.*  $\Phi_7$  and  $\Phi_8$  which consist of groups  $G$  such that  $|Z(G)| = p$  and  $cl(G) = 3$ . The family  $\Phi_7$  consists of groups  $G$  with  $G/Z(G) \approx \Phi_2(1^4)$ . Thus  $G/Z(G)$  is of exponent  $p$ . Since  $|\gamma_2(G)| = p^2$ , it follows that  $Z(G) < \gamma_2(G)$  and  $G/\gamma_2(G)$  is an elementary abelian group of order  $p^3$ . Thus  $d(G) = 3$  and hence  $\text{Aut}_z(G) \neq Z(\text{Inn}(G))$  by Theorem 3.2. The family  $\Phi_8$  consists of only one group *viz.*

$$\Phi_8(32) = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta = \alpha_1^p, \beta^{p^2} = \alpha_2^{p^2} = 1 \rangle$$

which is of rank 2 because  $\beta = [\alpha_1, \alpha_2] \in \gamma_2(\Phi_8(32)) \leq \Phi(\Phi_8(32))$  and hence  $\text{Aut}_z(\Phi_8(32)) = Z(\text{Inn}(\Phi_8(32)))$  by Theorem 3.2.

**Theorem 4.2** *If  $|G| = p^6$  and  $cl(G) = 3$  or 4, then  $\text{Aut}_z(G) = Z(\text{Inn}(G))$  if and only if  $G$  is isomorphic to one of the groups in the isoclinism families  $\Phi_{25}, \Phi_{26}, \Phi_{28}, \Phi_{29}$  and  $\Phi_{40} - \Phi_{43}$ .*

**Proof.** There are 16 isoclinism families *viz.*  $\Phi_{22}, \Phi_{24} - \Phi_{34}$  and  $\Phi_{40} - \Phi_{43}$  which consist of groups  $G$  for which  $|Z(G)| = p$  and  $cl(G) = 3$  or 4. If  $G$  is any group from the families  $\Phi_{22}, \Phi_{24}, \Phi_{27}$  and  $\Phi_{30} - \Phi_{33}$ , then  $Z(G) < \gamma_2(G)$ ,  $|\gamma_2(G)| = p^2$  or  $p^3$  and  $G/Z(G)$  is of exponent  $p$ . Therefore  $G$  is of rank 3 or 4 and hence  $\text{Aut}_z(G) \neq Z(\text{Inn}(G))$  by Theorem 3.3. The family  $\Phi_{34}$  consists of groups  $G$  such that  $|\gamma_2(G)| = p^3$ . From the presentations of these groups, it follows that they are generated by 6 elements  $\alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$  such that  $\beta_1, \beta_2, \gamma, \alpha^p, \alpha_1^p, \alpha_2^p \in \gamma_2(G)$ . This implies that  $\text{U}_1(G) \leq \gamma_2(G)$ . Thus  $d(G) = 3$  and hence  $\text{Aut}_z(G) \neq Z(\text{Inn}(G))$  by Theorem 3.3. Any group  $G$  of the families  $\Phi_{25}, \Phi_{26}, \Phi_{28}$  and  $\Phi_{29}$  is generated by 5 elements  $\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4$  such that  $\alpha_2, \alpha_3, \alpha_4 \in \gamma_2(G)$ . Thus  $d(G) = 2$  and hence  $\text{Aut}_z(G) = Z(\text{Inn}(G))$  by Theorem 3.3. If  $G$  is any group from the families  $\Phi_{40} - \Phi_{43}$ , then  $|\gamma_2(G)| = p^4$  and hence  $d(G) = 2$ . Thus  $\text{Aut}_z(G) = Z(\text{Inn}(G))$  by Theorem 3.3.

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